

# Degeneracies when only $T = 1$ two-body interactions are present

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## Abstract

In the nuclear  $f_{7/2}$  shell, the nucleon–nucleon interaction can be represented by the eight values  $E(J) = \langle (f_{7/2}^2)^J | V | (f_{7/2}^2)^J \rangle$ ,  $J = 0, 1, \dots, 7$ , where for even  $J$  the isospin is 1, and for odd  $J$  it is 0. If we set the  $T = 0$  (odd  $J$ ) two-body matrix elements to zero (or to a constant), we find several degeneracies which we attempt to explain in this work. We also give more detailed expressions than previously for the energies of the states in question. New methods are used to explain degeneracies that are found in  $^{45}\text{Ti}$  ( $I = 25/2^-$  and  $27/2^-$ ),  $^{46}\text{V}$  ( $I = 12_1^+$  and  $13_1^+$ , as well as  $I = 13_2^+$  and  $15^+$ ), and  $^{47}\text{V}$  ( $I = 29/2^-$  and  $31/2^-$ ).

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## I. INTRODUCTION

If we perform nuclear structure calculations in the  $f_{7/2}$  shell for systems of both valence neutrons and valence protons, our effective interaction consists of eight values  $E(J) = \langle (f_{7/2}^2)^J | V | (f_{7/2}^2)^J \rangle$ ,  $J = 0, 1, \dots, 7$ . The even  $J$  states have isospin  $T = 1$ , that is to say, they are isotriplets. With an isospin conserving interaction, one gets identical even  $J$  spectra for  $^{42}\text{Ca}$ ,  $^{42}\text{Sc}$ , and  $^{42}\text{Ti}$ . The odd  $J$  states have isospin  $T = 0$ ; they can only exist in  $^{42}\text{Sc}$ , the neutron-proton system.

Having chosen a set of  $E(J)$ , one can perform calculations of the spectra of more complicated nuclei, e.g., the Ti isotopes—2 protons and  $n$  neutrons, where  $n$  can range from 0 to 8. A not unreasonable choice is to equate  $E(J)$  with the yrast spectra of the two-particle system  $^{42}\text{Sc}$ .

In a previous work, we examined the behaviour of the spectra when all the  $T = 0$  two-body matrix elements were set to zero. We could also set them to a constant. This would not change the relative spectra of states with a given isospin, but it would change the relative energies of states of different isospin. In this work, we are always considering states with the lowest isospin, and so, nothing will be affected. We found various degeneracies with this simplified interaction. For example, in  $^{43}\text{Sc}$  ( $^{43}\text{Ti}$ ) the  $J = 13/2^-$  and  $1/2^-$  states were degenerate as were the  $J = 17/2^-$  and  $19/2^-$  states. In  $^{44}\text{Ti}$  the  $3_2^+$ ,  $7_2^+$ ,  $9_1^+$ , and  $10_1^+$  states were degenerate.

A common thread was found—that the states which were degenerate had angular momenta which could not occur for a system of identical particles, e.g.,  $J = 1/2^-, 13/2^-, 17/2^-$ , and  $19/2^-$  cannot occur in the  $f_{7/2}^3$  configuration of  $^{43}\text{Ca}$ . Likewise,  $3^+, 7^+, 9^+$ , and  $10^+$  cannot occur for the  $(f_{7/2}^4)$  configuration of  $^{44}\text{Ca}$ . Furthermore, it was found that these states displayed a partial dynamical symmetry that the angular momenta  $(J_P, J_N)$  were good dual quantum numbers.

An important point to be made is that, for the above mentioned nuclei, when one uses the full interaction (both  $T = 0$  and  $T = 1$  two-body matrix elements), we are not so far away from the limit where  $[J_P, J_N]$  are good quantum numbers (see Refs. [1, 2]). For example, using two-body matrix elements obtained from the spectrum of  $^{42}\text{Sc}$  (soon to be discussed), we find the following wave functions

$^{43}\text{Sc}$	$I = 13/2^-$	$0.98921[7/2, 4] + 0.14647[7/2, 6]$
$^{44}\text{Ti}$	$I = 3_2^+$	$0.12161([2, 4] - [4, 2]) - 0.69657([4, 6] - [6, 4])$
	$I = 7_2^+$	$0.13503([2, 6] - [6, 2]) - 0.69409([4, 6] - [6, 4])$
	$I = 9_1^+$	$-0.70711([4, 6] - [6, 4]) = -\frac{1}{\sqrt{2}}([4, 6] - [6, 4])$
	$I = 10_1^+$	$0.70089([4, 6] + [6, 4]) + 0.13234[6, 6]$

In  $^{43}\text{Sc}$  we are close to the limit  $[7/2, 4]$   $I = 13/2_1$ , and in  $^{44}\text{Ti}$ , to  $1/\sqrt{2}([4, 6] + (-1)^I[6, 4])$ . So, studying the limit where the  $T = 0$  matrix elements are set to zero makes sense.

We here note that there are other degeneracies present and that they require a different explanation. Here is the remaining list:

$$\begin{aligned}
^{45}\text{Ti} \quad I = 25/2^-, 27/2^- & \quad (T = 1/2) \\
^{46}\text{V} \quad I = 12_1^+, 13_1^+, \text{ and } 13_2^+, 15^+ & \quad (T = 0) \\
^{47}\text{V} \quad I = 29/2^-, 31/2^- & \quad (T = 1/2)
\end{aligned}$$

In the next sections, we will shed as much light as we can on these cases. In the calculations to be presented, we will use two interactions:  $V(^{42}\text{Sc})$  and  $T0V(^{42}\text{Sc})$ . The  $V(^{42}\text{Sc})$  interaction consists of the set of  $E(J)$ 's obtained by equating the latter to the excitation energy of the lowest state of angular momentum  $J$  in  $^{42}\text{Sc}$ ; the experimental values for  $J = 0$  to 7 are (in MeV) 0.0, 0.6111, 1.5863, 1.4904, 2.8153, 1.5101, 3.2420, and 0.6163, respectively. And the  $T0V(^{42}\text{Sc})$  interaction has the same values of  $E(J)$  for even  $J$  ( $T = 1$ ), but the values of  $E(J)$  for odd  $J$  ( $T = 0$ ) are set to zero.

By studying the nuclei and energy levels above, we are focusing on situations where the  $T = 0$  two-body matrix elements play an important role. In general, the effects of the  $T = 0$  matrix elements in nuclei are more elusive than those for  $T = 1$ . Two identical particles must have isospin  $T = 1$ , so that when we study, say, the tin isotopes in a model space involving only valence neutrons, the only two-body matrix elements are those with  $T = 1$ . And indeed the BCS theory in nuclei only involves  $T = 1$  matrix elements. We must seize whatever opportunity there is to study the effects of  $T = 0$  two-body matrix elements and we have, therefore, focused on cases which optimize this possibility.

## II. EXPLANATION OF THE DEGENERACIES IN $^{47}\text{V}$

We here address the degeneracies of  $I = 29/2^-$  and  $31/2^-$  states in  $^{47}\text{V}$ . Both these states are made by coupling an  $L_p = 15/2$  three-proton state to an  $L_n = 8$  four-neutron state. Therefore, they both have the same expectation values of the proton–proton interaction, and of the neutron–neutron interaction. However, since the proton and neutron components are coupled to different *total* angular momenta,  $29/2$  and  $31/2$ , it is not obvious that the proton–neutron interaction should have the same expectation value in both states. We will now show that this is indeed true in the absence of  $T = 0$  interactions. The same technique can be used for  $I = 25/2^-$  and  $27/2^-$  states of  $^{45}\text{Ti}$ .

It is convenient to start with wave functions in the  $m$  representation. Let  $[m_1 m_2 m_3]$  symbolize the normalized Slater determinant built out of the states  $\phi_{m_1}^j, \phi_{m_2}^j, \phi_{m_3}^j$ . A  $^{47}\text{V}$  state would be:  $[m_1 m_2 m_3]_\pi [n_1 n_2 n_3 n_4]_\nu$ , where the  $m_i$  set stands for the valence protons and the  $n_i$  set for the valence neutrons. The derivation here is quite general and can be applied not only to the  $f_{7/2}$  shell, but to other shells as well.

The proton–proton interaction would be

$$\begin{aligned} \langle [m_1 m_2 m_3] | \sum_{i < j=1}^3 V(i, j) | [m_1 m_2 m_3] \rangle &= \langle m_1 m_2 | V(1, 2) | m_1 m_2 - m_2 m_1 \rangle \\ &+ \langle m_1 m_3 | V(1, 2) | m_1 m_3 - m_3 m_1 \rangle \\ &+ \langle m_2 m_3 | V(1, 2) | m_2 m_3 - m_3 m_2 \rangle. \end{aligned} \quad (1)$$

And similarly for the neutron–neutron interaction.

The proton–neutron interaction would be

$$\langle [m_1 m_2 m_3] [n_1 n_2 n_3 n_4] | \sum_{i=1}^Z \sum_{j=1}^N V(i, j) | [m_1 m_2 m_3] [n_1 n_2 n_3 n_4] \rangle = \sum_{i=1}^Z \sum_{j=1}^N \langle m_i n_j | V(1, 2) | m_i n_j \rangle. \quad (2)$$

If  $V(1, 2)$  acts only in  $T = 1$  states, this can be written

$$\frac{1}{2} \sum_{i=1}^Z \sum_{j=1}^N \langle m_i n_j | V(1, 2) | m_i n_j - n_j m_i \rangle. \quad (3)$$

Now consider a state of the form  $[m_1 m_2 m_3] [m_1 m_2 m_3 m_4]$ . The essential point is that every proton state has a neutron partner and there is one extra neutron state ( $m_4$ ). Then

we have

$$\begin{aligned}
M^{pp+nn} &= \langle [m_1 m_2 m_3][m_1 m_2 m_3 m_4] | V^{pp} + V^{nn} | [m_1 m_2 m_3][m_1 m_2 m_3 m_4] \rangle \\
&= \langle m_1 m_2 | V | m_1 m_2 - m_2 m_1 \rangle + \langle m_1 m_3 | V | m_1 m_3 - m_3 m_1 \rangle + \langle m_2 m_3 | V | m_2 m_3 - m_3 m_2 \rangle \quad (4a) \\
&+ \langle m_1 m_2 | V | m_1 m_2 - m_2 m_1 \rangle + \langle m_1 m_3 | V | m_1 m_3 - m_3 m_1 \rangle + \langle m_2 m_3 | V | m_2 m_3 - m_3 m_2 \rangle \quad (4b) \\
&+ \langle m_1 m_4 | V | m_1 m_4 - m_4 m_1 \rangle + \langle m_2 m_4 | V | m_2 m_4 - m_4 m_2 \rangle + \langle m_3 m_4 | V | m_3 m_4 - m_4 m_3 \rangle \quad (4c)
\end{aligned}$$

where (4a) corresponds to the proton–proton interaction and (4b)–(4c) is the result of the neutron–neutron interaction. For the proton–neutron interaction, we have

$$\begin{aligned}
M^{pn} &= \langle [m_1 m_2 m_3][m_1 m_2 m_3 m_4] | V^{pn} | [m_1 m_2 m_3][m_1 m_2 m_3 m_4] \rangle \quad (5) \\
&= \frac{1}{2} \left[ \langle m_1 m_2 | V | m_1 m_2 - m_2 m_1 \rangle + \langle m_1 m_3 | V | m_1 m_3 - m_3 m_1 \rangle + \langle m_1 m_4 | V | m_1 m_4 - m_4 m_1 \rangle \right. \\
&\quad + \langle m_2 m_1 | V | m_2 m_1 - m_1 m_2 \rangle + \langle m_2 m_3 | V | m_2 m_3 - m_3 m_2 \rangle + \langle m_2 m_4 | V | m_2 m_4 - m_4 m_2 \rangle \\
&\quad \left. + \langle m_3 m_1 | V | m_3 m_1 - m_1 m_3 \rangle + \langle m_3 m_2 | V | m_3 m_2 - m_2 m_3 \rangle + \langle m_3 m_4 | V | m_3 m_4 - m_4 m_3 \rangle \right].
\end{aligned}$$

Comparing (4) and (5), we see that

$$M^{pn} = \frac{1}{2} M^{pp+nn}. \quad (6)$$

Thus, the expectation value of  $V^{pp} + V^{nn} + V^{pn}$  is  $\frac{3}{2} \times$  (the expectation value of  $V^{pp} + V^{nn}$ ).

It should be stressed that Eq. (6) has only been demonstrated for expectation values in states of the form  $[m_1 m_2 m_3][m_1 m_2 m_3 m_4]$ . Nevertheless, if  $T = 0$  interactions are absent, we can use this rather limited result to demonstrate the equality of  $M^{pn}$  in the  $I = 29/2^-$  and  $I = 31/2^-$  states of  $^{47}\text{V}$ . Analogous results hold for  $^{43}\text{Sc}$  ( $m_1[m_1 m_2]$ ),  $^{45}\text{Ti}$  ( $[m_1 m_2][m_1 m_2 m_3]$ ),  $^{49}\text{Cr}$  ( $[m_1 m_2 m_3 m_4][m_1 m_2 m_3 m_4 m_5]$ ), etc.

The  $J = M = 31/2$  state of  $^{47}\text{V}$  is  $\Psi_{31/2}^{31/2} = [\frac{7}{2} \frac{5}{2} \frac{3}{2}][\frac{7}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2}]$ . Apply the  $J_-$  operator to get the  $J = 31/2$ ,  $M = 29/2$  state:

$$\Psi_{29/2}^{31/2} = \sqrt{\frac{15}{31}} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} + \sqrt{\frac{16}{31}} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} - \frac{1}{2}. \quad (7)$$

The orthogonal combination is

$$\Psi_{29/2}^{29/2} = \sqrt{\frac{16}{31}} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{1}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} - \sqrt{\frac{15}{31}} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} \begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} - \frac{1}{2}. \quad (8)$$

Thus,

$$\begin{aligned}
& \langle \Psi_{29/2}^{31/2} | V^{pp} + V^{nn} + V^{pn} | \Psi_{29/2}^{31/2} \rangle = \\
& = \frac{15}{31} \left\langle \left[ \frac{7}{2} \frac{5}{2} \frac{1}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2} \right] \middle| V^{pp} + V^{nn} + V^{pn} \middle| \left[ \frac{7}{2} \frac{5}{2} \frac{1}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2} \right] \right\rangle \\
& + \frac{16}{31} \left\langle \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} - \frac{1}{2} \right] \middle| V^{pp} + V^{nn} + V^{pn} \middle| \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} - \frac{1}{2} \right] \right\rangle \\
& + 2 \frac{\sqrt{15 \cdot 16}}{31} \left\langle \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} - \frac{1}{2} \right] \middle| V^{pp} + V^{nn} + V^{pn} \middle| \left[ \frac{7}{2} \frac{5}{2} \frac{1}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2} \right] \right\rangle. \quad (9)
\end{aligned}$$

The last matrix element reduces to  $\langle \frac{3}{2}, -\frac{1}{2} | V^{pp} + V^{nn} + V^{pn} | \frac{1}{2} \frac{1}{2} \rangle$ , which is zero because  $(\frac{1}{2} \frac{1}{2})$  is a  $T = 0$  state and  $V^{pp} + V^{nn} + V^{pn}$  only act in  $T = 1$  states. Alternatively, the exchange term cancels the direct term because  $m_n = m_p = 1/2$ .

Thus

$$\begin{aligned}
& \langle \Psi_{29/2}^{31/2} | V^{pp} + V^{nn} + V^{pn} | \Psi_{29/2}^{31/2} \rangle = \\
& = \frac{15}{31} \left\langle \left[ \frac{7}{2} \frac{5}{2} \frac{1}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2} \right] \middle| V^{pp} + V^{nn} + V^{pn} \middle| \left[ \frac{7}{2} \frac{5}{2} \frac{1}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2} \right] \right\rangle \\
& + \frac{16}{31} \left\langle \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} - \frac{1}{2} \right] \middle| V^{pp} + V^{nn} + V^{pn} \middle| \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} \right] \left[ \frac{7}{2} \frac{5}{2} \frac{3}{2} - \frac{1}{2} \right] \right\rangle. \quad (10)
\end{aligned}$$

Each of these matrix elements is of the form

$$\langle [m_1 m_2 m_3] [m_1 m_2 m_3 m_4] | V^{pp} + V^{nn} + V^{pn} | [m_1 m_2 m_3] [m_1 m_2 m_3 m_4] \rangle, \quad (11)$$

and our previous work shows that the expectation value of  $V^{pn}$  is half of the expectation value of  $V^{pp} + V^{nn}$ . Hence,

$$\langle \Psi_{29/2}^{31/2} | V^{pp} + V^{nn} + V^{pn} | \Psi_{29/2}^{31/2} \rangle = \frac{3}{2} \langle \Psi_{29/2}^{31/2} | V^{pp} + V^{nn} | \Psi_{29/2}^{31/2} \rangle. \quad (12)$$

Similarly

$$\langle \Psi_{29/2}^{29/2} | V^{pp} + V^{nn} + V^{pn} | \Psi_{29/2}^{29/2} \rangle = \frac{3}{2} \langle \Psi_{29/2}^{29/2} | V^{pp} + V^{nn} | \Psi_{29/2}^{29/2} \rangle. \quad (13)$$

But

$$\begin{aligned}
& \langle \Psi_{29/2}^{J=29/2,31/2} | V^{pp} + V^{nn} | \Psi_{29/2}^{J=29/2,31/2} \rangle = \\
& = \langle [\psi^{15/2}(1, 2, 3) \psi^8(1, 2, 3, 4)]_{29/2}^J | V^{pp} + V^{nn} | [\psi^{15/2}(1, 2, 3) \psi^8(1, 2, 3, 4)]_{29/2}^J \rangle \\
& = \langle \psi_M^{15/2}(1, 2, 3) | V^{pp} | \psi_M^{15/2}(1, 2, 3) \rangle + \langle \psi_M^8(1, 2, 3, 4) | V^{nn} | \psi_M^8(1, 2, 3, 4) \rangle, \quad (14)
\end{aligned}$$

independent of  $J$ . Therefore

$$\langle \Psi_{29/2}^{31/2} | V^{pp} + V^{nn} + V^{pn} | \Psi_{29/2}^{31/2} \rangle = \langle \Psi_{29/2}^{29/2} | V^{pp} + V^{nn} + V^{pn} | \Psi_{29/2}^{29/2} \rangle. \quad (15)$$

Thus, we have proved that when only  $T = 1$  two-body matrix elements are present, the  $J = 29/2^-$  and  $31/2^-$  states in  $^{47}\text{V}$  are degenerate.

Similar equalities as the one in Eq. (15) exist for the  $J = 19/2, 17/2$  states of  $^{43}\text{Sc}$ , and the  $J = 27/2, 25/2$  states of  $^{45}\text{Ti}$ .

In the  $g_{9/2}$  shell, for a system of 1 proton and 2 neutrons (analog of  $^{43}\text{Sc}$  in the  $f_{7/2}$  shell), the  $J = 19/2^-, 23/2^-$ , and  $25/2^-$  states are degenerate. For a system of 2 protons and 3 neutrons (in analogy with  $^{45}\text{Ti}$ ), we find that the  $J = 35/2^-$  and  $37/2^-$  states are degenerate with  $T=0V(^{42}\text{Sc})$ . Also, for a system of 3 protons and 4 neutrons (the analog of  $^{47}\text{V}$ ), the states with  $J = 43/2^-$  and  $45/2^-$  are degenerate.

### III. DETAILED EXPRESSIONS FOR THE ENERGIES IN $^{43}\text{Sc}$ ( $^{43}\text{Ti}$ ), $^{44}\text{Ti}$ , $^{45}\text{Ti}$ , AND $^{46}\text{V}$

Throughout this section, the numbers that we will give for the energy differences are calculated using the interaction  $V(^{42}\text{Sc})$ .

#### A. $^{43}\text{Sc}$ ( $^{43}\text{Ti}$ )

In a previous work, it was noted that when the  $T = 0$  two-body matrix elements were set equal to a constant (which might as well be 0), there were certain degeneracies and, for selected states in  $^{43}\text{Sc}$ ,  $(J_P, J_N)$  were good quantum numbers. The selected states were those with angular momenta  $I$  which could not be found in  $^{43}\text{Ca}$ , i.e.,  $1/2, 13/2, 17/2$ , and  $19/2$ , all of them with isospin  $T = 1/2$ . For these states, we have a partial dynamical symmetry, but for the other angular momenta which can occur in  $^{43}\text{Ca}$  ( $3/2, 5/2, 7/2, 9/2, 11/2$ , and  $15/2$ ) we do not have such a symmetry.

Consider first the  $I = 17/2^-$  and  $19/2^-$  states. The basic configuration is  $(J_P = 7/2, J_N = 6)$ . Since the wave function is antisymmetric in the two neutrons, the expectation values of  $V(p, n_1)$  and  $V(p, n_2)$  are the same. Thus we have:

$$\begin{aligned}
& \langle [j6]^I | V | [j6]^I \rangle \\
& \equiv \langle \{ \psi^j(p) [\psi^j(n_1) \psi^j(n_2)]^6 \}_M^I | V(n_1, n_2) + V(p, n_1) + V(p, n_2) | \{ \psi^j(p) [\psi^j(n_1) \psi^j(n_2)]^6 \}_M^I \rangle \\
& = E(6) + 2 \langle \{ \psi^j(p) [\psi^j(n_1) \psi^j(n_2)]^6 \}_M^I | V(p, n_1) | \{ \psi^j(p) [\psi^j(n_1) \psi^j(n_2)]^6 \}_M^I \rangle. \tag{16}
\end{aligned}$$

To evaluate the expectation value of  $V(p, n_1)$ , we re-couple the three-particle states in the bra and ket,

$$\{\psi^j(p)[\psi^j(n_1)\psi^j(n_2)]^6\}_M^I = \sum_{I_x} U(jjIj; I_x 6) \{[\psi^j(p)\psi^j(n_1)]^{I_x}\psi^j(n_2)\}_M^I, \quad (17)$$

where we have used Jahn's notation[3] for the unitary 6- $j$  recoupling amplitude

$$\begin{aligned} U(j_1 j_2 I j_3; j_{12} j_{23}) &= \langle \{[j_1 j_2]^{j_{12}} j_3\}_M^I \mid \{j_1 [j_2 j_3]^{j_{23}}\}_M^I \rangle \\ &= (-1)^{j_1+j_2+I+j_3} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & I & j_{23} \end{Bmatrix}, \end{aligned} \quad (18)$$

where the last factor is the usual 6- $j$  symbol. Thus

$$\begin{aligned} \langle [j6]^I | V | [j6]^I \rangle &= E(6) + 2 \sum_{I_x} (U(jjIj; I_x 6))^2 \times \\ &\quad \times \langle \{[\psi^j(p)\psi^j(n_1)]^{I_x}\psi^j(n_2)\}_M^I | V(p, n_1) | \{[\psi^j(p)\psi^j(n_1)]^{I_x}\psi^j(n_2)\}_M^I \rangle \\ &= E(6) + 2 \sum_{I_x} (U(jjIj; I_x 6))^2 E(I_x). \end{aligned} \quad (19)$$

$$(20)$$

For  $I = 17/2$  and  $19/2$  we have

$$\begin{aligned} E(17/2^-) &= 0.73077E(5) + 1.5E(6) + 0.76923E(7), \\ E(19/2^-) &= 1.5E(6) + 1.5E(7). \end{aligned}$$

Hence, the difference in energy is given by

$$E(17/2^-) - E(19/2^-) = 0.73077[E(5) - E(7)]. \quad (21)$$

This depends only on the  $T = 0$  two-body matrix elements and it vanishes when they are set equal to a constant. Using  $V(^{42}\text{Sc})$ , we get a value of 0.65316 MeV for Eq. (21).

As a slightly more complex example, consider the  $J = 13/2^-$  states in  $^{43}\text{Sc}$ . They are linear combinations of the basis states  $(J_P, J_N)$   $[7/2, 4]$  and  $[7/2, 6]$ . The coupling matrix element is

$$\begin{aligned} \langle [j4]^{13/2} | V | [j6]^{13/2} \rangle &= 2 \sum_{I_x} U(jj13/2j; 4I_x) U(jj13/2j; 6I_x) E(I_x) \\ &= 0.32909E(3) - 0.55710E(5) + 0.22791E(7), \end{aligned} \quad (22)$$

which is equal to  $-0.21034$  MeV. Note that this coupling matrix element depends only on the  $T = 0$  two-body matrix elements and vanishes when they are set equal to a constant.

Hence, when the  $T = 0$  two-body matrix elements are set equal to a constant, the two eigenfunctions for  $I = 13/2^-$  become  $[7/2, 4]$  and  $[7/2, 6]$ . In that limit, the expressions for the energies of the  $[j, 4]$  configurations for  $I = 13/2$  and  $1/2$  become

$$\langle [\frac{7}{2}, 4]^{13/2} | V | [\frac{7}{2}, 4]^{13/2} \rangle = 0.144628E(3) + 1.5E(4) + 0.447552E(5) + 0.907819E(7) \quad (23)$$

$$\langle [\frac{7}{2}, 4]^{1/2} | V | [\frac{7}{2}, 4]^{1/2} \rangle = 1.5E(3) + 1.5E(4). \quad (24)$$

Although the two expressions involve both  $T = 0$  and  $T = 1$  two-body matrix elements, the difference  $E(13/2) - E(1/2)$  depends only on  $T = 0$  two-body matrix elements and vanishes if they are set equal to a constant.

We next find

$$\langle [7/2, 6]^{13/2} | V | [7/2, 6]^{13/2} \rangle = 0.749311E(3) + 0.693473E(5) + 1.5E(6) + 0.057215E(7). \quad (25)$$

Note that when  $E(3)$ ,  $E(5)$ , and  $E(7)$  are set to a constant, we get this state to be degenerate with the  $I = 17/2^-$  and  $19/2^-$  states, which also have the configuration  $[7/2, 6]$ .

When a full diagonalization is performed for the  $I = 13/2$  states, the degeneracies with the other states are removed. The lowest  $I = 13/2$  state with a dominant  $[j, 4]$  configuration is 0.81583 MeV below the  $I = 1/2$  state, while the other  $I = 13/2$  state is 1.30592 MeV above the  $19/2$  state. In this “full interaction” case, the above energy differences involve both  $T = 0$  and  $T = 1$  two-body matrix elements.

We were able to understand the above results by noting certain properties of  $6j$  symbols and further by providing physical explanation for these properties. First we give the mathematical results.

The lack of coupling for  $I = 13/2$  of the two configurations  $[7/2, 4]$  and  $[7/2, 6]$  could be explained by noting the following property of a  $6j$  symbol

$$\left\{ \begin{matrix} 7/2 & 7/2 & 4 \\ 7/2 & 13/2 & 6 \end{matrix} \right\} = 0, \quad (26)$$

which can be generalized to

$$\left\{ \begin{matrix} j & j & (2j-3) \\ j & (3j-4) & (2j-1) \end{matrix} \right\} = 0 \quad (27)$$

and applied to other shells.

To explain the degeneracies of  $13/2^-$ ,  $17/2^-$ , and  $19/2^-$ , all with the configuration  $[j, 6]$ , we note

$$\begin{Bmatrix} j & j & (2j-1) \\ j & I & (2j-1) \end{Bmatrix} = \frac{-1}{8j-2} \quad (28)$$

for  $I = 13/2, 17/2$ , and  $19/2$  (but not for  $I = 15/2$ ).

To explain the degeneracy of  $I = 1/2^-$  and  $13/2^-$  with the  $[j, 4]$  configuration, we have that  $\begin{Bmatrix} 7/2 & 7/2 & 4 \\ 7/2 & I & 4 \end{Bmatrix}$  is the same for  $I = 1/2$  and  $13/2$ , and its value is 0.055555.

The results of Eqs. (27) and (28) can be understood physically by following arguments of Racah [4, 5] and de Shalit and Talmi [6]. The vanishing of the first  $6j$  above [Eq. (27)] can be explained by trying to construct a cfp to a state which is forbidden by the Pauli principle, e.g., to an  $I = 13/2^-$  state in  $^{43}\text{Ca}$ . One constructs a cfp by the principal parent method  $(j^2(J_1)j|j^3I[J_0])$ , coupling first two identical particles to an even  $J_0$ . Then one can form the wave function as

$$N(1 - P_{13} - P_{23}) [(j^2)^{J_0}j]^I \quad (29)$$

and rewrite it as

$$\sum_{J_1} (j^2(J_1)j|j^3I[J_0]) [(j^2)^{J_1}j]^I. \quad (30)$$

By choosing the principal parent  $J_0$  to be  $2j-3$  and taking  $J_1 = 2j-1$ , one gets the condition of Eq. (27). By choosing  $J_0 = 2j-1$  and  $J_1 = 2j-1$ , one gets the condition of Eq. (28). More details are given in the works of Robinson and Zamick [1, 2] and will not be repeated here. Amusingly, for  $j = 7/2$ , when one constructs cfp's to allowed states, then one gets the same cfp (or zero) no matter what principal parent one chooses. This is because each allowed state occurs only once ( $I = 3/2, 5/2, 7/2, 9/2, 11/2, 15/2$ ). However, when one calculates cfp's to forbidden states, one gets new useful information for each choice of a principal parent.

## B. $^{44}\text{Ti}$

It was noted by Robinson and Zamick [1, 2] that there were also several degeneracies in  $^{44}\text{Ti}$  when the  $T = 0$  two-body matrix elements were set to a constant. In that case,  $(J_P, J_N)$  became good dual quantum numbers for select states and there were degeneracies.

For example, the states  $I = 3_2^+, 7_2^+, 9_1^+$ , and  $10_1^+$ , with the configuration

$$\frac{1}{\sqrt{2}} \{ [4, 6] + (-1)^I [6, 4] \}, \quad (31)$$

were degenerate. It was pointed out that the above angular momenta could not appear in  $^{44}\text{Ca}$  and, if we attempted to construct two-particle cfp's to these forbidden states, then those cfp's must vanish. By choosing different principal parents, the authors obtained decoupling conditions (to make  $(J_P, J_N)$  good dual quantum numbers) and the degeneracy condition.

The decoupling conditions are

$$\begin{Bmatrix} 7/2 & 7/2 & 4 \\ 7/2 & 7/2 & 4 \\ 6 & 4 & I \end{Bmatrix} = 0 \quad (32)$$

for  $I = 3$  and  $7$ , and

$$\begin{Bmatrix} 7/2 & 7/2 & 6 \\ 7/2 & 7/2 & 6 \\ 6 & 4 & I \end{Bmatrix} = 0 \quad (33)$$

for  $I = 3, 7, 9$ , and  $10$ . The generalization of the latter condition for other  $j$  shells is

$$\begin{Bmatrix} j & j & (2j-1) \\ j & j & (2j-1) \\ (2j-1) & (2j-3) & I \end{Bmatrix} = 0 \quad (34)$$

for  $I = 2j - 4$  and  $I = 4j - 4$ .

In Shadow Robinson's 2002 thesis [7], there are two degeneracy conditions. First we have

$$\begin{Bmatrix} j & j & (2j-3) \\ j & j & (2j-1) \\ (2j-3) & (2j-1) & I \end{Bmatrix} = \frac{1}{4(4j-5)(4j-1)}, \quad (35)$$

which is independent of  $I$ , but only for certain values of  $I$ . For example, for  $j = 7/2$ ,  $I = 3, 7, 9$ , and  $10$ . None of these angular momenta can occur for the  $f_{7/2}^4$  configuration of identical particles. In the  $g_{9/2}$  shell, Eq. (35) holds for  $I = 11, 13$ , and  $14$ , which are the only angular momenta that cannot occur for a system of four identical particles in the  $g_{9/2}$  shell.

A second condition is [7]

$$\left\{ \begin{array}{ccc} j & j & (2j-1) \\ j & j & (2j-1) \\ (2j-1) & (2j-1) & I \end{array} \right\} = \frac{1}{2(4j-1)^2} \quad (36)$$

for  $I = (4j-4)$  and  $(4j-2)$ . For these two values, the result is independent of  $I$ . In the  $f_{7/2}$  shell, this applies to  $I = 10$  and  $12$ . In the  $g_{9/2}$  shell, it applies to  $I = 14$  and  $16$ .

Recently, Zhao and Arima have derived these results [7] in a different way considering systems of four identical particles [8].

The expressions for the respective energies of the degenerate configurations shown in Eq. (31) are as follows

$$\begin{aligned} I = 3_2^+ & : 0.722222E(1) + 1.449495E(3) + 1.5E(4) + 0.777778E(5) + 1.5E(6) + 0.050505E(7) \\ I = 7_2^+ & : 0.173469E(1) + 1.179653E(3) + 1.5E(4) + 0.681842E(5) + 1.5E(6) + 0.965035E(7) \\ I = 9_1^+ & : 0.333333E(3) + 1.5E(4) + 1.166667E(5) + 1.5E(6) + 1.500000E(7) \\ I = 10_1^+ & : 0.154270E(3) + 1.5E(4) + 0.700465E(5) + 1.5E(6) + 2.145264E(7), \end{aligned}$$

and the energy differences are

$$\begin{aligned} E(3_2^+) - E(7_2^+) & = 0.548753E(1) + 0.269842E(3) + 0.095936E(5) - 0.914530E(7) \\ E(3_2^+) - E(9_1^+) & = 0.722222E(1) + 1.116162E(3) - 0.388889E(5) - 1.449495E(7) \\ E(3_2^+) - E(10_1^+) & = 0.722222E(1) + 1.295225E(3) + 0.077313E(5) - 2.094759E(7) \\ E(7_2^+) - E(9_1^+) & = 0.173469E(1) + 0.846320E(3) - 0.484825E(5) - 0.534965E(7) \\ E(7_2^+) - E(10_1^+) & = 0.173469E(1) + 1.025383E(3) - 0.018623E(5) - 1.180229E(7) \\ E(9_1^+) - E(10_1^+) & = 0.179063E(3) + 0.466202E(5) - 0.645264E(7) \end{aligned}$$

We can readily see that all these differences depend only on the  $T = 0$  two-body matrix elements and that they vanish when the said matrix elements are all set equal to a constant.

### C. $^{45}\text{Ti}$ and $^{47}\text{V}$

In  $^{45}\text{Ti}$  the configuration for the states with total angular momentum  $I = 25/2^-$ ,  $27/2^-$  and isospin  $T = 1/2$  is  $(J_P = 6, J_N = 15/2)$ , while for  $^{47}\text{V}$  the configuration for  $I = 29/2^-$ ,  $31/2^-$  with  $T = 1/2$  is  $(J_P = 15/2, J_N = 8)$ . Note that, as in the cases of  $^{43}\text{Sc}$  and  $^{44}\text{Ti}$ ,  $(J_P, J_N)$  are good quantum numbers.

It will turn out that although this is a necessary condition for degeneracy, it is not sufficient. For example, in  $^{45}\text{Sc}$  there are two states with unique  $J_P$  and  $J_N$ , namely ( $J_P = 7/2, J_N = 8$ ),  $I = 21/2^-$  and  $23/2^-$ . However, they are *not* degenerate when the  $T = 0$  two-body matrix elements are set to zero. Another example is  $^{46}\text{Ti}$  ( $J_P = 6, J_N = 8$ ),  $I = 13$  and  $14$ . These also are not degenerate in that limit.

We now give detailed expressions for the excitation energies of the states in  $^{45}\text{Ti}$  and  $^{47}\text{V}$  that have been discussed above, in terms of the  $E(J)$ 's .

For the unique configuration  $[6, 15/2]$  of  $^{45}\text{Ti}$ , we have

$$I = \frac{25}{2}^- : 0.545454E(3) + 1.022727E(4) + 0.942307E(5) + 4.977272E(6) + 2.512236E(7)$$

$$I = \frac{27}{2}^- : 1.022727E(4) + 0.826923E(5) + 4.977272E(6) + 3.173074E(7)$$

The differences involve only the  $T = 0$  two-body matrix elements

$$E(25/2^-) - E(27/2^-) = 0.545454E(3) + 0.115384E(5) - 0.660838E(7). \quad (37)$$

Using  $V(^{42}\text{Sc})$ , we obtain 0.57992 MeV for this difference. Note that Eq. (37) vanishes if  $E(3)$ ,  $E(5)$ , and  $E(7)$  are set equal.

For  $^{47}\text{V}$  the corresponding results for the unique configuration  $[15/2, 8]$  are

$$\begin{aligned} I = \frac{29}{2}^- : & 0.369048E(1) + 0.714286E(2) + 0.810606E(3) + 3.535714E(4) \\ & + 1.996337E(5) + 9.249999E(6) + 4.324010E(7) \\ I = \frac{31}{2}^- : & 0.714286E(2) + 0.575758E(3) + 3.535714E(4) + 1.967949E(5) \\ & + 9.249999E(6) + 4.956295E(7) \end{aligned}$$

$$E(29/2^-) - E(31/2^-) = 0.369048E(1) + 0.234848E(3) + 0.028388E(5) - 0.632285E(7), \quad (38)$$

which is equal to 0.22873 MeV.

#### IV. $^{46}\text{V}$

In  $^{46}\text{V}$  the  $I = 12^+$  and  $13^+$  states (both have isospin  $T = 0$ ) are degenerate when the two-body  $T = 0$  matrix elements are set to a constant. In that limit, the states in question

have the following structures

$$I = 12 : \frac{1}{\sqrt{2}} [(j^3)_\pi^{15/2} (j^3)_\nu^{11/2} - (j^3)_\pi^{11/2} (j^3)_\nu^{15/2}] \quad (39a)$$

$$I = 13 : \frac{1}{\sqrt{2}} [(j^3)_\pi^{15/2} (j^3)_\nu^{11/2} + (j^3)_\pi^{11/2} (j^3)_\nu^{15/2}]. \quad (39b)$$

Note that  $J_P$  and  $J_N$  are good dual quantum numbers, just as they were in the other cases ( $^{43}\text{Sc}$ ,  $^{44}\text{Ti}$ ,  $^{45}\text{Ti}$ , and  $^{47}\text{V}$ ). The nucleus  $^{46}\text{V}$  is the only case where the angular momenta in question (12 and 13) *can* occur in  $^{46}\text{Ca}$ .

The expressions for the energies of these states are

$$\begin{aligned} & E[(j^3)^{15/2}] + E[(j^3)^{11/2}] \\ & + \sum_{J_0 J'_0} \sum_{I_x I_y} \left\{ 9(j^2 J_0 j | \{ j^3 15/2 \})^2 (j^2 J'_0 j | \{ j^3 11/2 \})^2 [ \langle (J_0 j)^{15/2} (J'_0 j)^{11/2} | (J_0 J'_0)^{I_y} (j j)^{I_x} \rangle^{12} ]^2 E(I_x) \right. \\ & - 9(j^2 J_0 j | \{ j^3 15/2 \}) (j^2 J_0 j | \{ j^3 11/2 \}) (j^2 J'_0 j | \{ j^3 15/2 \}) (j^2 J'_0 j | \{ j^3 11/2 \}) \\ & \times \langle (J_0 j)^{15/2} (J'_0 j)^{11/2} | (J_0 J'_0)^{I_y} (j j)^{I_x} \rangle^{12} \langle (J_0 j)^{11/2} (J'_0 j)^{15/2} | (J_0 J'_0)^{I_y} (j j)^{I_x} \rangle^{12} E(I_x) \left. \right\} \quad (40) \end{aligned}$$

for  $I = 12$ .

For  $I = 13$ , we have

$$\left\langle \frac{15}{2} \frac{11}{2} \middle| V \middle| \frac{15}{2} \frac{11}{2} + \frac{11}{2} \frac{15}{2} \right\rangle = C + D, \quad (41)$$

where

$$\begin{aligned} C = & E[(j^3)^{15/2}] + E[(j^3)^{11/2}] + \sum_{J_0 J'_0} \sum_{I_x I_y} 9(j^2 J_0 j | \{ j^3 15/2 \})^2 (j^2 J_0 j | \{ j^3 11/2 \})^2 \\ & \times [ \langle (J_0 j)^{15/2} (J'_0 j)^{11/2} | (J_0 J'_0)^{I_y} (j j)^{I_x} \rangle^{13} ]^2 E(I_x), \quad (42a) \end{aligned}$$

$$\begin{aligned} D = & \sum_{J_0 J'_0} \sum_{I_x I_y} 9(j^2 J_0 j | \{ j^3 15/2 \}) (j^2 J'_0 j | \{ j^3 11/2 \}) \langle (J_0 j)^{15/2} (J'_0 j)^{11/2} | (J_0 J'_0)^{I_y} (j j)^{I_x} \rangle^{13} \\ & \times \langle (J_0 j)^{11/2} (J'_0 j)^{15/2} | (J_0 J'_0)^{I_y} (j j)^{I_x} \rangle^{13} E(I_x). \quad (42b) \end{aligned}$$

In Tables I and II, we present results of a single  $j$  shell calculation of the wave function of the  $I = 0, 12, 13, 14$ , and  $15$  states of  $^{46}\text{V}$  [9]. For Table I the calculations have been made with the  $V(^{42}\text{Sc})$  interaction, while in Table II we have used  $T0V(^{42}\text{Sc})$ . The excitation energies are shown in the first rows. The wave functions are represented as column vectors  $D^{I\alpha}(J_P, J_N)$ , where  $D$  is the probability amplitude that, for the  $\alpha$ -th state of total angular momentum  $I$ , the protons couple to  $J_P$  and the neutrons to  $J_N$ .

TABLE I: Energy levels and wave functions of selected states of  $^{46}\text{V}$  with  $V(^{42}\text{Sc})$  [9].

$I = 0$							
		0.00000	4.62474	6.27338	7.89321	9.31823	13.20357
$J_P$	$J_N$						
1.5	1.5	0.22825	0.26657	-0.89465	-0.09857	-0.12933	0.22361
2.5	2.5	0.56868	-0.77447	-0.02361	0.02048	0.02853	0.27386
3.5	3.5	0.69821	0.28981	0.09263	-0.04876	0.13250	-0.63246
4.5	4.5	0.21580	0.23830	0.35363	-0.65916	-0.46054	0.35355
5.5	5.5	0.27863	0.34419	0.22811	0.72974	-0.26333	0.38730
7.5	7.5	0.11314	0.26437	0.11566	-0.14307	0.82672	0.44721
$I = 12$							
		7.96301	8.02569	8.31797	9.99248	10.85364	
$J_P$	$J_N$	$T = 0$	$T = 1$	$T = 0$	$T = 1$	$T = 1$	
4.5	7.5	0.54264	0.21115	0.45337	0.60498	-0.29902	
5.5	7.5	0.45337	0.62253	-0.54264	-0.29557	-0.15840	
7.5	4.5	0.54264	-0.21116	0.45337	-0.60498	0.29902	
7.5	5.5	-0.45337	0.62253	0.54264	-0.29557	-0.15840	
7.5	7.5	0.00000	0.36842	0.00000	0.30540	0.87806	
$I = 13$							
		7.09970	9.86810	10.23589			
$J_P$	$J_N$	$T = 0$	$T = 0$	$T = 1$			
5.5	7.5	0.70314	-0.07476	-0.70711			
7.5	5.5	0.70314	-0.07476	0.70711			
7.5	7.5	0.10573	0.99440	0.00000			
$I = 14$							
		10.52667					
$J_P$	$J_N$						
7.5	7.5	1.00000					
$I = 15$							
		9.05871					
$J_P$	$J_N$						
7.5	7.5	1.00000					

Looking first at Table I (full interaction), we see some striking visual effects which are fairly easy to explain. Note that the amplitude  $D^I(J_P, J_N)$  is either the same or of opposite sign to  $D^I(J_N, J_P)$ , which is a consequence of charge symmetry. One can show that  $D^{I,T}(J_P, J_N) = (-1)^{J_P+J_N-I+T} D^{I,T}(J_N, J_P)$ ; and this explains why the amplitude  $D^{I=12,T=0}(7.5, 7.5) = 0$ .

Another amusing fact is that the numerical coefficients for the two  $I = 12, T = 0$  states

TABLE II: Energy levels and wave functions of selected states of  $^{46}\text{V}$  with  $T0V(^{42}\text{Sc})$ .

$I = 0$							
		0.00000	3.64508	5.63620	5.98779	7.21330	12.67985
$J_P$	$J_N$						
1.5	1.5	0.20195	0.16385	-0.82144	-0.34819	-0.29388	0.22361
2.5	2.5	0.40942	-0.84466	0.06891	-0.19087	0.05232	0.27386
3.5	3.5	0.75303	0.16921	0.03292	-0.01398	0.05515	-0.63246
4.5	4.5	0.24927	0.29219	0.53563	-0.22854	-0.62318	0.35355
5.5	5.5	0.33161	0.08173	-0.14034	0.84386	-0.03941	0.38730
7.5	7.5	0.22900	0.37284	0.11316	-0.27892	0.71968	0.44721
$I = 12$							
		7.26601	7.69828	8.02036	9.67282	10.33978	
$J_P$	$J_N$	$T = 0$	$T = 0$	$T = 1$	$T = 1$	$T = 1$	
4.5	7.5	0.00000	$1/\sqrt{2}$	0.21708	0.56219	0.36989	
5.5	7.5	$-1/\sqrt{2}$	0.00000	0.56957	-0.36048	0.21363	
7.5	4.5	0.00000	$1/\sqrt{2}$	-0.21708	-0.56219	-0.36989	
7.5	5.5	$1/\sqrt{2}$	0.00000	0.56958	-0.36048	0.21363	
7.5	7.5	0.00000	0.00000	0.50688	0.32862	-0.79692	
$I = 13$							
		7.26601	9.27745	9.92755			
$J_P$	$J_N$	$T = 0$	$T = 0$	$T = 1$			
5.5	7.5	$1/\sqrt{2}$	0.00000	$-1/\sqrt{2}$			
7.5	5.5	$1/\sqrt{2}$	-0.00000	$1/\sqrt{2}$			
7.5	7.5	0.00000	1.00000	0.00000			
$I = 14$							
		10.59802					
$J_P$	$J_N$						
7.5	7.5	1.00000					
$I = 15$							
		9.27745					
$J_P$	$J_N$						
7.5	7.5	1.00000					

are the same but occur for different values of  $J_P, J_N$  (see  $I = 12$  states with energies 7.96302 and 8.31797 MeV in Table I). This can also be easily explained by the combination of charge symmetry and the fact that the two wave functions must be orthogonal. It would be of interest, experimentally, to study the two lowest  $T = 0, I = 11$  states, e.g. by looking at  $E2$  and  $M1$  transitions from  $11_2$  to  $11_1$ .

We now come to Table II, in which the  $T = 0$  two-body matrix elements were set to zero.

There are no special points of interest for the  $I = 0$  states, except to note that the wave function of the highest energy state does not change. This is because it is a unique  $T = 3$  state, an analog of the  $I = 0$  state in  $^{46}\text{Ca}$ .

However, for  $I = 12, 13$ , and  $15$ , we see several points of interest. The lowest  $I = 12$  ( $T = 0$ ) state is degenerate with the lowest  $I = 13$  ( $T = 0$ ) state. Also the two  $I = 12$ ,  $T = 0$  states have a simple structure:  $(0, -1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$  and  $(1/\sqrt{2}, 0, 1/\sqrt{2}, 0, 0)$ , respectively. For  $I = 13$  the first wave function has the simple structure  $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ , while the second one is  $(0, 0, 1)$ . The latter state is degenerate with the unique  $I = 15$  state.

These results are more difficult to explain than what we just discussed before.

### A. The $I = 13$ state

We here address why the second  $I = 13$  state has the simple structure shown above when the  $T = 0$  two-body matrix elements are set equal to zero. We again go to the  $m$  representation and construct all the  $M = 13$  states for  $T = 0$ . These are

$$|A\rangle = \frac{1}{\sqrt{2}} \left\{ \left[ \begin{smallmatrix} 7 & 5 \\ 2 & 2 \end{smallmatrix} - \frac{1}{2} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 5 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} + \left[ \begin{smallmatrix} 7 & 5 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 5 \\ 2 & 2 \end{smallmatrix} - \frac{1}{2} \right]_{\nu} \right\} \quad (43a)$$

$$|B\rangle = \frac{1}{\sqrt{2}} \left\{ \left[ \begin{smallmatrix} 7 & 3 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 5 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} + \left[ \begin{smallmatrix} 7 & 5 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 3 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} \right\} \quad (43b)$$

$$|C\rangle = \left[ \begin{smallmatrix} 7 & 5 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 5 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} \quad (43c)$$

We will now show that there is no coupling between states  $C$  and  $B$ , and between  $C$  and  $A$  when there are only  $T = 1$  two-body matrix elements present.

$$\begin{aligned} M_{CB} &= \frac{1}{\sqrt{2}} \left\langle \left[ \begin{smallmatrix} 7 & 5 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 5 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} \left| V \right| \left[ \begin{smallmatrix} 7 & 3 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 5 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} + \left[ \begin{smallmatrix} 7 & 5 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\pi} \left[ \begin{smallmatrix} 7 & 3 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \right]_{\nu} \right\rangle \\ &= \left\langle \frac{(5/2)_{\pi}(1/2)_{\nu} + (1/2)_{\pi}(5/2)_{\nu}}{\sqrt{2}} \left| V \right| \left( \frac{3}{2} \right)_{\pi} \left( \frac{3}{2} \right)_{\nu} \right\rangle. \end{aligned} \quad (44)$$

But  $(3/2)_{\pi}(3/2)_{\nu}$  is necessarily a  $T = 0$  state (space symmetric) and hence the matrix element vanishes when only  $T = 1$  interactions are present.

Similar results hold for the  $C$ - $A$  coupling

$$M_{CA} = \left\langle \left( \frac{1}{2} \right)_{\pi} \left( \frac{1}{2} \right)_{\nu} \left| V \right| \frac{(-1/2)_{\pi}(3/2)_{\nu} + (3/2)_{\pi}(-1/2)_{\nu}}{\sqrt{2}} \right\rangle = 0. \quad (45)$$

Thus, the  $m$  scheme wave function  $\left[\frac{7}{2}\frac{5}{2}\frac{1}{2}\right]_{\pi} \left[\frac{7}{2}\frac{5}{2}\frac{1}{2}\right]_{\nu}$  is an eigenfunction of the Hamiltonian in which only  $T = 1$  interactions are present. Now we can write

$$\left[\frac{7}{2}\frac{5}{2}\frac{1}{2}\right]_{\pi} \left[\frac{7}{2}\frac{5}{2}\frac{1}{2}\right]_{\nu} = a \left[\frac{15}{2}\frac{15}{2}\right]_{M=13}^{I=15} + b \left[\frac{15}{2}\frac{15}{2}\right]_{M=13}^{I=13}. \quad (46)$$

(One cannot have  $J_P$  or  $J_N$  equal to  $11/2$  or lower because  $M_P = 13/2$ ,  $M_N = 13/2$ . Besides, the  $I = 14$  state has isospin  $T = 1$ .)

Since we have an eigenfunction of  $H$ , we can write

$$(H - E) \left( a \left[\frac{15}{2}\frac{15}{2}\right]_{M=13}^{I=15} + b \left[\frac{15}{2}\frac{15}{2}\right]_{M=13}^{I=13} \right) = 0. \quad (47)$$

But, since  $[15/2, 15/2]^{15}$  is a unique configuration, it is also an eigenfunction of  $H$  with eigenvalue  $E'$ . We get

$$a(E' - E) \left[\frac{15}{2}\frac{15}{2}\right]_{M=13}^{I=15} + b(H - E) \left[\frac{15}{2}\frac{15}{2}\right]_{M=13}^{I=13} = 0. \quad (48)$$

Operating to the left with the bra  $\left\langle \left[\frac{15}{2}\frac{15}{2}\right]_{13}^{15} \right|$ , it yields

$$a(E' - E) + b(E' - E) \left\langle \left[\frac{15}{2}\frac{15}{2}\right]^{15} \middle| \left[\frac{15}{2}\frac{15}{2}\right]^{13} \right\rangle = 0. \quad (49)$$

The expectation value in the second term is obviously zero. Hence, we find  $E' = E$ . This implies:  $(H - E)[15/2, 15/2]^{13} = 0$ . We have, thus, proved that the  $I = 15$  and  $I = 13$  states are degenerate and that the wave function of the  $I = 13$  state in question is  $[J_P = 15/2, J_N = 15/2]^{13}$ .

## B. The $I = 12$ state

For the  $I = 12$  state, we will be briefer. The degenerate  $I = 12$  and  $13$  states with  $M = 12$  in  $^{46}\text{V}$  are

$$A = \frac{1}{\sqrt{2}} \left[ \Psi_{15/2}^{15/2}(\pi) \Psi_{9/2}^{11/2}(\nu) + \Psi_{15/2}^{15/2}(\nu) \Psi_{9/2}^{11/2}(\pi) \right] \quad (50a)$$

$$B = \frac{1}{\sqrt{2}} \left[ \Psi_{13/2}^{15/2}(\pi) \Psi_{11/2}^{11/2}(\nu) + \Psi_{13/2}^{15/2}(\nu) \Psi_{11/2}^{11/2}(\pi) \right], \quad (50b)$$

with coefficients  $(15/2, 11/2, 15/2, 9/2|J, 12)$  and  $(15/2, 11/2, 13/2, 11/2|J, 13)$  for  $I = 12$  and  $13$ , respectively.

We can show that, if only  $T = 1$  matrix elements are present, there is no matrix element between  $A$  and  $B$  and the state

$$C = \frac{1}{\sqrt{2}} \left[ \Psi_{15/2}^{15/2}(\pi) \Psi_{9/2}^{9/2}(\nu) - \Psi_{15/2}^{15/2}(\nu) \Psi_{9/2}^{9/2}(\pi) \right]. \quad (51)$$

Unfortunately, the proof is rather detailed. After throwing away all  $\langle mm|V|m_1m_2 \rangle$  and  $\langle m_1m_2|V|mm \rangle$  matrix elements, we are left with the non-vanishing matrix elements  $\langle \frac{1}{2} - \frac{1}{2}|V|\frac{3}{2} - \frac{3}{2} \rangle$  and  $\langle \frac{1}{2}\frac{7}{2}|V|\frac{5}{2}\frac{3}{2} \rangle$ . However, each comes in twice, with opposite signs, so they cancel. Details will be omitted.

Using Eqs. (40–42) for the case when the  $T = 0$  matrix elements are set equal to zero, we find the following expressions for the energies of the degenerate  $[15/2, 11/2]^{I=12,13}$  configurations

$$\begin{aligned} I = 12 : & 0.060606E(1) + 1.250000E(2) + 0.626033E(3) + 1.909091E(4) \\ & + 2.391608E(5) + 5.840909E(6) + 2.921752E(7) \\ I = 13 : & 0.153061E(1) + 1.250000E(2) + 0.587662E(3) + 1.909091E(4) \\ & + 1.264521E(5) + 5.840909E(6) + 3.994755E(7) \end{aligned}$$

These involve  $T = 1$  matrix elements, but they get cancelled out when we take the energy difference. Note also the absence of an  $E(0)$  term. Of course, we have set  $E(0)$  to zero; but, even if we did not, the coefficient of  $E(0)$  would still be zero. The reason is that, if we have a proton and a neutron coupled to  $(J = 0, T = 1)$ , then the remaining two protons and two neutrons would have to be coupled to  $(J = 12, T = 1)$  in order to get a final result of  $(I = 12, T = 0)$ . But there is no  $(J = 12, T = 1)$  state in  $^{44}\text{Ti}$  in the single  $j$  shell model. Likewise, there are no  $J = 13$  states in  $^{44}\text{Ti}$  in the single  $j$  shell approximation.

## V. FULL INTERACTION RESULTS IN THE $f_{7/2}$ SHELL MODEL SPACE

In Table III we present the excitation energies of those states that were degenerate when the  $T = 0$  two-body matrix elements were set equal to zero. We thus get a feeling for the effects of the  $T = 0$  matrix elements in the calculation. For example, for  $^{45}\text{Ti}$  we obtain

$$E(25/2^-) - E(27/2^-) = 8.46781 - 7.88789 = 0.57992 \text{ MeV}.$$

TABLE III: Calculation of energies in the  $f_{7/2}$ -shell-model space using the full  $V(^{42}\text{Sc})$  interaction for states that are degenerate when the  $T = 0$  two-body matrix elements are set equal to zero.

Nucleus	Leading configuration $[J_P, J_N]$	$J$	$T$	$E$
$^{43}\text{Sc}$	$[7/2, 4]$	$1/2_1$	$1/2$	4.31596
		$13/2_1$	$1/2$	3.50013
	$[7/2, 6]$	$13/2_2$	$1/2$	4.95078
		$17/2$	$1/2$	4.29802
		$19/2$	$1/2$	3.64486
$^{44}\text{Ti}$	$[4, 6] \pm [6, 4]$	$3_2$	0	8.69411
		$7_2$	0	8.37435
		$9_1$	0	7.98380
		$10_1$	0	7.38394
	$[6, 6]$	$10_2$	0	8.90568
		$12_1$	0	7.70224
$^{45}\text{Ti}$	$[6, 15/2]$	$25/2$	$1/2$	8.46781
		$27/2$	$1/2$	7.88789
$^{46}\text{V}^a$	$[15/2, 11/2] \pm [11/2, 15/2]$	$12_1$	0	7.96302
		$12_2$	0	8.31797
		$13_1$	0	7.09970
	$[15/2, 15/2]$	$13_2$	0	9.86809
		$15_1$	0	9.05871

<sup>a</sup>In  $^{46}\text{V}$  there is strong mixing of the various  $[J_P, J_N]$  configurations, so we list both  $J = 12$ ,  $T = 0$  states. They also have substantial  $[15/2, 9/2]$  mixing.

## VI. CLOSING REMARKS

In previous works [1, 2], we have studied degeneracies in single- $j$ -shell calculations for  $^{43}\text{Sc}$  and  $^{44}\text{Ti}$  when  $T = 0$  two-body matrix elements are set equal to zero. In this work, we give some detailed expressions for the energies, which were not present before. But the main thrust of this work is to handle degeneracies in  $^{45}\text{Ti}$ ,  $^{46}\text{V}$ , and  $^{47}\text{V}$ , which also occur

in this limit, but are more difficult to explain. For these nuclei, we use a different approach by switching to the  $m$  scheme. For a given nucleus, we consider only states of the lowest possible isospin  $T = |N - Z|/2$ . The new degeneracies were  $I = 25/2^-$  and  $27/2^-$  in  $^{45}\text{Ti}$ ;  $I = 29/2^-$  and  $31/2^-$  in  $^{47}\text{V}$ ; and  $I = 12_1^+$  and  $13_1^+$ , as well as  $13_2^+$  and  $15^+$ , in  $^{46}\text{V}$ .

A common feature that emerges is that, for all cases considered, the degenerate states have  $[J_P, J_N]$  as good dual quantum numbers. We have shown this to be true on a case-by-case basis. One often associates degeneracy with a symmetry. But, even though we have explained all the degeneracies, we have not found a symmetry associated with them and there might well not be one. We have, however, noted a common feature—in all nuclei considered, the angular momenta for which degeneracies are present cannot occur for systems of identical particles in a single  $j$  shell.

It should be noted that there are some states that have  $[J_P, J_N]$  as good dual quantum numbers, but are *not* degenerate in the limit of only  $T = 1$  matrix elements being present.

Lastly, we reiterate that we have pointed out cases of experimental interest for those involved in studying the effects of the  $T = 0$  interaction in a nucleus.

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